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THE REVOLUTION OF A DARK PARTICLE ABOUT A LUMINOUS CENTER

By Edwin Bidwell Wilson

1. Introduction. The simple problem of two bodies moving under their mutual gravitational attraction has doubtless long been considered as completely solved. Yet ever since Maxwell's formulation of his theory of electromagnetic action as a motion or stress of some kind in the ether, there has been a remote possibility that the time might come for revising the dynamical equations of motion so as to take into account the action of the ether on the moving masses. In fact, recent investigators of the electronic theories of matter have paid a great deal of attention to the interaction of matter and ether. To adapt their work to the problem of the revolution of bodies about a fixed luminous center, which is approximately the case that obtains in the solar system, is by no means easy. For in the first place there are different theories of electrons and in the second place these theories lead one to try to replace the older and more familiar dynamical equations by newer and as yet not fully settled electrodynamical equations. Thus Abraham* discusses in considerable detail the actions of the ether on a moving body; but his equations are those of the electromagnetic field and depend to some extent on his particular theory.

Recently, however, Poynting, with that directness of insight and sharp mechanical instinct which characterize the English investigators of physical science, has taken up the problem of the effect of radiation on moving masses and has treated it from the point of view of the transmission of momentum by the ether. His results, although perhaps essentially electromagnetic in that they are derived from the consideration of waves in the ether, are nevertheless well adapted to direct mechanical formulation.† The effects which he describes are three:[*] the direct pressure of light, the Doppler reception effect, and the Doppler emission effect. We will state what is meant by each.

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* *Theorie der Elektrizität*, vol. 2.

† We do not wish to be interpreted as implying either that the pressure of radiation is *necessarily* connected with any electromagnetic theory or that Poynting was the first to consider the matter from other points of view. On the contrary the question is rather one of mechanics and thermo dynamics and had previously been treated as such by several authors. Historical references may be found in an article on "The pressure due to radiation" by Nichols and Hull, *Astrophysical Journal*, vol. 17, June, 1903 or *The Smithsonian Report for* 1903, pp. 115–138. Poynting merely reverts to this simpler treatment and applies it anew.

If a body which is under the rays of the sun be considered as dark[†] or essentially black, to use the technical term, the radiation from the sun is stopped and absorbed by the body. The result is an outward pressure due to the abstraction of momentum from the ether. As the electromagnetic radiation varies in its intensity according to the same law as the gravitational attraction, it produces a pressure which varies inversely as the square of the distance and hence introduces no change in the dynamical equations except an apparently diminished value for the gravitational action. It might seem at first sight that in estimating the pressure due to radiation the distance should be measured from the radiating surface: but a moment's consideration shows that a spherical body like the sun could be replaced by a concentric body radiating with a properly altered intensity, and hence that the distance should be measured from the center.

The two Doppler effects are much smaller than the direct pressure of light. To consider the reception effect suppose that the dark body were moving toward the sun or any other luminous center with a velocity v. Then a greater amount of momentum would be absorbed per unit time than if the body were at rest. In particular the amount would be greater in the ratio v/V, where V denotes the velocity of light. In case the body falls to the sun from an infinite distance this ratio is about 2/1000, which may be considered as its maximum value in the solar system; generally it is much smaller. If the body retreats, the pressure is lessened in the same ratio. Finally there is the emission effect which arises from the fact that the dark body radiates the energy it receives. This force, which is "frictional" in that it always opposes the motion, is stated by Poynting to be one-third of v/V of the direct pressure of light.* This does not make the emission effect merely one-third as great as the reception effect: for the velocity which enters into the determination of the reception effect is the component velocity of the motion resolved along the line joining the particle and the sun, whereas in the emission effect it is the total velocity in path that enters into the formula.

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[*] See his address to the Royal Society, *Nature*, vol. 75, pp. 90–93, November 22, 1906. Earlier than this, Poynting had printed an extensive article on "Radiation in the solar system: its effect on temperature and its pressure on small bodies," *Transactions of the Royal Society*, ser. A, vol. 202 (1903), pp. 525–552 In this article there occur some minor errors which the author corrects in a note at the end. Moreover the two Doppler effects are by no means clearly distinguished one from another. Finally in this earlier treatment, *loc. cit.*, p. 549, the author comes upon a differential equation that he states is probably not integrable This equation corresponds to our equation (5). The integration which we give in terms of Bessel functions not only furnishes a desirable complete determination of the problem, but enables us to see certain other general characteristics of the motion than are obtainable from a less exact integration.

[†] This is approximately true for the small particles in the solar system. As far as we know them, their albedo is very low.

2. The differential equations of motion and their integrals. These statements of Poynting's may be incorporated in the dynamical equations in a simple manner if polar coordinates with the sun as center be assumed for reference. Let $-C/r^2$ and K/r^2 respectively be the accelerations due to the gravitational attraction and the direct pressure of radiation. There will be the accelerations

$$-\frac{K}{r^2}\frac{1}{V}\frac{dr}{dt} \quad \text{and} \quad -\frac{1}{3}\frac{K}{r^2}\frac{1}{V}\frac{dr}{dt}$$

along the radius vector due respectively to the reception and emission effects. Further there will be the single acceleration

$$-\frac{1}{3}\frac{K}{r^2}\frac{1}{V}\frac{rd\theta}{dt}$$

perpendicular to the radius. The equations of motion then become †

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{C-K}{r^2} - \frac{4}{3}\frac{K}{Vr^2}\frac{dr}{dt}, \tag{1}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -\frac{1}{3}\frac{K}{Vr^2}\frac{rd\theta}{dt}. \tag{2}$$

The simplicity of these equations is largely due to the fortunate linear dependence of the Doppler effects on the velocity of the particle. Equation (2) is immediately integrable into

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* As a matter of fact Poynting states, as is well known, that higher powers of v/V enter into the expression; but neglecting them does not introduce, on the assumptions made above, an error of more than one-fifth of one per cent in a quantity already small; and generally the error is much less than this. To introduce the exact expression for the emission effect (which is evaluated on assumptions that are not perfectly well established) would unnecessarily complicate the equations of motion.

† Provided we disregard, as we may, any changes in the mass of the particle which might arise from its motion through the electromagnetic field. On the electronic theories there is such an effect, but it remains very small until the velocity of the moving body is within a few percent of the velocity of light.

$$r^2 \frac{d\theta}{dt} = h - \frac{K}{3V}\theta. \tag{3}$$

To find the orbit of the particle we may eliminate the time between this equation and (1). Meanwhile in order to reduce the resulting equation to a simpler form, it will be well to introduce the change of variables

$$u = \frac{1}{r}, \quad \frac{3Vh}{K} - \theta = \phi. \tag{4}$$

The differential equation of the orbit then takes the particularly simple form

$$\frac{d^2u}{d\phi^2} - \frac{3}{\phi}\frac{du}{d\phi} + u = \frac{9V^2}{K^2}\frac{C-K}{\phi^2} = \frac{L}{\phi^2}. \tag{5}$$

Two things may be noted at this point. First, if K be allowed to approach zero, ϕ becomes infinite in such a manner that $K\phi$ remains finite and $d\phi = -d\theta$. Hence (5) approaches, as it should, the form

$$\frac{d^2u}{d\theta^2} + u = \text{const.},$$

ordinarily obtained in the theory of gravitation for the revolution of a planet. Secondly it is noteworthy that (5) is, except for the presence of the second member, an equation of the Bessel type.

To bring the differential equation of the orbit into shape for easier comparison with the treatises on Bessel functions the change of variable $u = z\phi^2$ will be made. Then (5) takes the final form:

$$\frac{d^2z}{d\phi^2} + \frac{1}{\phi}\frac{dz}{d\phi} + \left(1 - \frac{2^2}{\phi^2}\right)z = \frac{L}{\phi^4}. \tag{6}$$

If the second member were zero, the general solution would be

$$z = AJ_2(\phi) + BY_2(\phi)$$

in terms of Bessel functions J and Neumann functions Y.* It remains to find a particular solution of the complete equation to add to this. These particular solutions, rarely given in the books, may readily be obtained by the method of variation of constants. In this case such a solution is

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* The notations throughout this article will conform to those of the *Treatise on Bessel Functions* by Gray and Mathews (1895).

$$L\left[Y_2(\phi)\int\frac{J_2(\phi)}{\phi^3}\,d\phi - J_2(\phi)\int\frac{Y_2(\phi)}{\phi^3}\,d\phi\right].$$

Hence the equation of the orbit may be written as

$$u = \phi^2\left\{AJ_2(\phi) + BY_2(\phi) + L\left[Y_2(\phi)\int\frac{J_2(\phi)}{\phi^3}\,d\phi - J_2(\phi)\int\frac{Y_2(\phi)}{\phi^3}\,d\phi\right]\right\},$$

where the constants A and B may be chosen so that the lower limit of integration may take any desired value.

As the Doppler effects are very small, the motion of the particle will be nearly elliptic, parabolic, or hyperbolic. To bring out more clearly the departures from these forms we may expand the solution (7) by means of the asymptotic expressions for the Bessel and Neumann functions. These series will semi-converge with extreme rapidity because of the large value of the argument ϕ (we shall have no occasion to consider cases in which ϕ is materially less than 10^6 as may be seen by reference to p. 148). For such cases as these the function Y is inconvenient: it contains an additive multiple of J. Let us write *

$$Y_n(\phi) = (\log 2 - \gamma)\,J_n(\phi) + Z_n\,(\phi).$$

Then the function Z satisfies the fundamental relations which J and Y satisfy and which are the reduction formulas† especially useful in the manipulation of these functions. We have then the asymptotic formulas

$$\left(\frac{\pi\phi}{2}\right)^{1/2}J_n(\phi) = P\cos\left\{\frac{(2n+1)\pi}{4} - \phi\right\} + Q\sin\left\{\frac{(2n+1)\pi}{4} - \phi\right\},$$

$$\left(\frac{2\phi}{\pi}\right)^{1/2}Z_n(\phi) = Q\cos\left\{\frac{(2n+1)\pi}{4} - \phi\right\} - P\sin\left\{\frac{(2n+1)\pi}{4} - \phi\right\},$$

where

$$P = 1 - \frac{(4n^2-1)(4n^2-9)}{2\,!(8\phi)^2} + \text{higher even powers of } \frac{1}{\phi},$$

$$Q = \frac{4n^2-1}{8\phi} - \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3\,!(8\phi)^3} + \text{higher odd powers of } \frac{1}{\phi}.$$

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* See Gray and Mathews, p. 40, formula 95.

† These are: $Z'_n = (n/\phi)Z_n - Z_{n+1} = Z_{n-1} - (n/\phi)Z_n$, $Z_{n+1} - (2n/\phi)Z_n + Z_{n-1} = 0$, $J_{n+1}Z_n - J_nZ_{n+1} = 0$, and $(\phi^nZ_n)' = \phi^nZ_{n-1}$. See Gray and Mathews, p. 14 and p. 16.

Or if we set $(2n+1)\ \pi/4 - \phi = \psi_n$ we may write

$$(8)\qquad \begin{aligned} \left(\frac{\pi\phi}{2}\right)^{1/2} J_n(\phi) &= \cos\psi_n + \frac{n^2-\frac{1}{4}}{2\phi}\sin\psi_n - \frac{(n^2-\frac{1}{4})(u^2-\frac{9}{4})}{2\,!(2\phi)^2}\cos\psi_n + \cdots, \\ \left(\frac{2\phi}{\pi}\right)^{1/2} Z_n(\phi) &= -\sin\psi_n + \frac{n^2-\frac{1}{4}}{2\phi}\cos\psi_n + \frac{(n^2-\frac{1}{4})(u^2-\frac{9}{4})}{2\,!(2\phi)^2}\sin\psi_n + \cdots, \end{aligned}$$

with the relations $\cos\psi_{n-1} = \sin\psi_n$, $\sin\psi_{n-1} = -\cos\psi_n$. The terms omitted from (8) are of the order 10^{-18} when ϕ is of the order 10^6.

In terms of the functions J and Z the equation of the orbit becomes

$$(9)\qquad u = \phi^2\left\{CJ_2(\phi) + DZ_2(\phi) + L\left[Z_2(\phi)\int\frac{J_2(\phi)}{\phi^3}d\phi - J_2(\phi)\int\frac{Z_2(\phi)}{\phi^3}d\phi\right]\right\},$$

where C and D are constants of integration slightly different from A and B. This form of the equation is not yet in the shape most convenient for computation. For the integrals in the bracket change their value by a hundred percent when ϕ runs through an interval of 2π. It may be noted, however, that the maximum value of the integrals, when extended over a small interval in ϕ, is very small owing to the fact that the denominator remains of the order 10^{18} and the numerator of order 10^{-3}. An integration by parts will enable us to accentuate this. We have

$$\int\frac{J_2(\phi)}{\phi^3}\,d\phi = \int\frac{\phi^3 J_2(\phi)}{\phi^6}\,d\phi = \frac{\phi^3 J_3(\phi)}{\phi^6} + 6\int\frac{\phi^3 J_3(\phi)}{\phi^7}\,d\phi = \frac{J_3(\phi)}{\phi^3} + 6\int\frac{\phi^4 J_3(\phi)}{\phi^8}\,d\phi.$$

Hence

$$\int\frac{J_2(\phi)}{\phi^2}\,d\phi = \frac{J_3(\phi)}{\phi^3} + 6\,\frac{J_4(\phi)}{\phi^4} + 6\cdot 8\,\frac{J_5(\phi)}{\phi^5} + \cdots,$$

with a similar result for the integral of $Z_2(\phi)/\phi^3$. Combining, we have for the bracket

$$Z_2(\phi)\int\frac{J_2(\phi)}{\phi^3}\,d\phi - J_2(\phi)\int\frac{Z_2(\phi)}{\phi^3}\,d\phi = \frac{1}{\phi^4} - \frac{12}{\phi^6} + \text{higher even powers of } \frac{1}{\phi}.$$

There appears to be no necessity of keeping terms of the order 10^{-48}. Hence

$$(10)\qquad u = \phi^2\left\{CJ_2(\phi) + DZ_2(\phi) + L\left(\frac{1}{\phi^4} - \frac{12}{\phi^6}\right)\right\}$$

may be taken as the approximate equation of the orbit for future computations. For the derivative we have, from (10), the expression

$$u' = \phi^2 \left\{ CJ_1(\phi) + DZ_1(\phi) - L\frac{2}{\phi^5} \right\}, \tag{11}$$

where the small quantity $48\,L/\phi^7$ has been neglected within the brace

If the initial values u_0, u'_0, and ϕ_0 of u, u', and ϕ be substituted in (10) and (11), the constants C and D may be obtained.

$$C = \frac{1}{\phi_0}\left\{ \left[u_0 - L\left(\frac{1}{\phi_0^2} - \frac{12}{\phi_0^4}\right)\right] Z_1(\phi_0) - \left[u'_0 + \frac{2L}{\phi_0^8}\right] Z_2(\phi_0) \right\}$$

$$= \frac{1}{\phi_0}\Big(RZ_1(\phi_0) - SZ_2(\phi_0)\Big),$$

$$D = \frac{1}{\phi_0}\left\{ -\left[u_0 - L\left(\frac{1}{\phi_0^2} - \frac{12}{\phi_0^4}\right)\right] J_1(\phi_0) + \left[u'_0 + \frac{2L}{\phi_0^3}\right] J_2(\phi_0) \right\}$$

$$= \frac{1}{\phi_0}\Big(-RJ_1(\phi_0) + SJ_2(\phi_0)\Big).$$

In equations (8) let $\psi_n = \psi_2 = \psi$. Then we have, if $(2n+1)\pi/4 - \phi_0 = \psi_0$,

$$\left(\frac{\pi\phi_0}{2}\right)^{1/2} J_1(\phi_0) = \sin\psi_0 - \frac{\frac{3}{4}}{2\phi_0}\cos\psi_0 - \frac{\frac{3}{4}\cdot -\frac{5}{4}}{2!(2\phi_0)^2}\sin\psi_0 + \cdots,$$

$$\left(\frac{2\phi_0}{\pi}\right)^{1/2} Z_1(\phi_0) = \cos\psi_0 + \frac{\frac{3}{4}}{2\phi_0}\sin\psi_0 - \frac{\frac{3}{4}\cdot -\frac{5}{4}}{2!(2\phi_0)^2}\cos\psi_0 + \cdots.$$

Hence

$$Z_1(\phi_0)J_2(\phi) = \left(\frac{1}{\phi_0\phi}\right)^{1/2}\Bigg[\cos\psi_0\cos\psi + \cos\psi_0\sin\psi\,\frac{\frac{15}{4}}{2\phi} + \sin\psi_0\cos\psi\,\frac{\frac{3}{4}}{2\phi_0}$$

$$+ \sin\psi_0\sin\psi\,\frac{\frac{3}{4}\cdot\frac{15}{4}}{4\phi_0\phi} - \cos\psi_0\cos\psi\,\frac{\frac{15}{4}\cdot\frac{7}{4}}{2!(2\phi)^2} - \cos\psi_0\cos\psi\,\frac{\frac{3}{4}\cdot -\frac{5}{4}}{2!(2\phi_0)^2}\Bigg],$$

and

$$J_1(\phi_0)Z_2(\phi) = \left(\frac{1}{\phi_0\phi}\right)^{1/2}\Bigg[-\sin\psi_0\sin\psi + \sin\psi_0\cos\psi\,\frac{\frac{15}{4}}{2\phi} + \cos\psi_0\sin\psi\,\frac{\frac{3}{4}}{2\phi_0}$$

$$- \cos\psi_0\cos\psi\,\frac{\frac{3}{4}\cdot\frac{15}{4}}{4\phi_0\phi} + \sin\psi_0\sin\psi\,\frac{\frac{15}{4}\cdot\frac{7}{4}}{2!(2\phi)^2} + \sin\psi_0\sin\psi\,\frac{\frac{3}{4}\cdot -\frac{5}{4}}{2!(2\phi_0)^2}\Bigg].$$

Hence

$$RZ_1(\phi_0)J_2(\phi) - RJ_1(\phi_0)Z_2(\phi) = R\left(\frac{1}{\phi_0\phi}\right)^{1/2}\Big[\cos(\psi_0 - \psi)$$
$$- \sin(\psi_0 - \psi)\left(\frac{15}{8\phi} - \frac{3}{8\phi_0}\right) + \cos(\psi_0 - \psi)\left(\frac{45}{64\phi_0\phi} - \frac{105}{128\phi^2} + \frac{15}{128\phi_0^2}\right)\Big].$$

In a similar manner

$$- SZ_2(\phi_0)J_2(\phi) + SJ_2(\phi_0)Z_2(\phi)$$
$$= S\left(\frac{1}{\phi_0\phi}\right)^{1/2}\Big[\sin(\psi_0 - \psi) + \cos(\psi_0 - \psi)\left(\frac{15}{8\phi} - \frac{15}{8\phi_0}\right)$$
$$+ \sin(\psi_0 - \psi)\left(\frac{225}{64\phi_0\phi} - \frac{105}{128\phi^2} - \frac{105}{128\phi_0^2}\right)\Big].$$

Now let $\psi_0 - \psi = \xi$. Then $\phi - \phi_0 = \xi$ and $\phi = \phi_0(1 + \xi/\phi_0)$. Hence, by expanding into series for two terms and collecting, we have

$$u = \left(\frac{\phi}{\phi_0}\right)^{3/2}\Big\{R\cos\xi + S\sin\xi - R\sin\xi\,\frac{3}{2\phi_0} + R\xi\sin\xi\,\frac{15}{8\phi_0^2} - S\xi\cos\xi\,\frac{15}{8\phi_0^2}$$
$$+ S\sin\xi\,\frac{240}{128\phi_0^2}\Big\} + L\phi^2\left(\frac{1}{\phi^4} - \frac{12}{\phi^6}\right).$$

It should next be noted that the coefficients R and S consist of the sum of terms of different orders of magnitude. In fact $u_0 - L/\phi_0^2$ may be considered as of the second order whereas $12L/\phi_0^4$ is of the fourth. So also in S we have u_0' of the second order (unless the orbit differs only infinitesimally from a circle, u' and u are of the same order) whereas $2L/\phi_0^3$ is of the third. If we rearrange the expression for u on this basis and at the same time expand $(\phi/\phi_0)^{3/2}$ we get as the final form.

$$u = \left[\left(u_0 - \frac{L}{\phi_0^2}\right)\cos\xi + u_0'\sin\xi + \frac{L}{\phi_0^2}\right]$$
$$(12) \qquad + \frac{1}{\phi_0}\left[\frac{1}{2}\left(\frac{7L}{\phi_0^2} - 3u_0\right)\sin\xi - \frac{2L}{\phi_0^2}\xi + \frac{3}{2}u_0'\xi\sin\xi + \frac{3}{2}\left(u_0 - \frac{L}{\phi_0^2}\right)\xi\cos\xi\right]$$
$$+ \frac{1}{\phi_0^2}\Big[\frac{15}{8}u_0'\sin\xi - \frac{24L}{\phi_0^2}\sin^2\tfrac{1}{2}\xi - \frac{15}{4}u_0'\xi\cos\xi - \frac{3}{8}\left(u_0 - \frac{9L}{\phi_0^2}\right)\xi\sin\xi$$
$$+ \frac{3L}{\phi_0^2}\xi^2 + \frac{3}{8}\left(u_0'\sin\xi + \left(u_0 - \frac{L}{\phi_0^2}\right)\cos\xi\right)\xi^2\Big],$$

and hence, on differentiating,

$$\begin{aligned} u' = & \left[u_0' \cos \xi - \left(u_0 - \frac{L}{\phi_0^2}\right) \sin \xi\right] \\ & + \frac{1}{\phi_0}\left[\frac{3}{2} u_0' \left(\sin \xi + \xi \cos \xi\right) - \frac{3}{2}\left(u_0 - \frac{L}{\phi_0^2}\right) \xi \sin \xi - \frac{4L}{\phi_0^2} \sin^2 \tfrac{1}{2} \xi\right]. \\ & + \frac{1}{\phi_0^2}\left[\frac{6L}{\phi_0^2} \xi - \frac{3}{8} u_0 \sin \xi - \frac{69}{8} \frac{L}{\phi_0^2} \sin \xi + \frac{21}{8} u_0' \xi \sin \xi \right. \\ & \left. + \frac{3}{8}\left(u_0 + \frac{7L}{\phi_0^2}\right) \xi \cos \xi + \frac{3}{8}\left(u_0' \cos \xi - \left(u_0 - \frac{L}{\phi_0^2}\right) \sin \xi\right) \xi^2\right], \end{aligned} \tag{13}$$

where the first, second, and third brackets denote respectively the first, second, and third approximations. The integration of the problem, both in exact and in approximate form, may therefore be considered as accomplished.

3. Discussion of the characteristics of the motion. In the first place let it be assumed that at the start the areal velocity is positive. Equation (3) shows that as the motion goes on and θ increases from its initial value 0, the areal velocity will steadily decrease and hence by equation (4) the argument ϕ will decrease. If we allow ϕ to approach zero in the complete integral (7), it appears that u is becoming infinite: for the terms

$$\phi^2 J_2(\phi), \quad \phi^2 Y_2(\phi), \quad \phi^2 J_2(\phi) \int \frac{Y_2(\phi)}{\phi^3} d\phi$$

approach finite limits and the integral*

$$\int \frac{J_2(\phi)}{\phi^3} d\phi$$

diverges for $\phi = 0$. Hence the particle will reach the attracting center when $\phi = 0$. It should be noted, however, that in an actual case the motion will be stopped before the particle reaches the center by the impact of the particle on the surface of the attracting mass. For another reason it is also valueless to investigate what happens to the orbit for very small values of ϕ, that is, when the particle is near the attracting center. We have assumed in setting up our equations that the velocity of the particle did not become very great.

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* This follows immediately on substituting the principal values of the functions obtainable from the developments in series. See Gray and Mathews, p. 11 and p. 14.

Hence the particle should not be allowed to approach the center too closely. Hence $3\,Vh/K$ is an upper limit for θ, and we may say that the number of revolutions that the particle can make is less than

$$N = \frac{3\,Vh}{2\pi K}. \tag{14}$$

This number for any given particle depends solely on the initial areal velocity and not on the velocity of projection or the initial distance from the center, except in so far as they conjointly determine the areal velocity.

The approximate solution (12) will serve to discuss the motion for a few dozen revolutions from the initial position $\phi = \phi_0$. If it be assumed that the polar axis is so chosen that $u_0' = 0$, that is, so that the particle starts from an apse, then the first approximation may be put in the form

$$u = \frac{L}{\phi_0^2}\,(1 + e\cos\xi), \qquad e = \frac{u_0 - \dfrac{L}{\phi_0^2}}{\dfrac{L}{\phi_0^2}}.$$

This is a conic section as might have been expected. In particular it is that conic section which the particle would described under the action of the gravitational attraction diminished by the action of the direct pressure of light and free from the Doppler effects. This conic I shall call the osculating conic of the orbit.

To find the amount by which the particle falls in from the apse toward the center during the first complete revolution it is merely necessary to subtract the values of the radius vector as given by (12) for $\xi = 0$ and $\xi = -2\pi$. Hence to the second approximation we have

$$\Delta u_0 = u_0\,\frac{1}{N}\,\frac{1}{1+e}\left(2 - \frac{3}{2}\,e\right), \qquad \Delta r_0 = -\,r_0\,\frac{1}{N}\,\frac{1}{1+e}\left(2 - \frac{3}{2}\,e\right).$$

This quantity $\Delta\, r_0$ is always negative, showing an approach to the center in all cases,* and varies between the limits $r_0/4N$ and $2r_0/N$ as the osculating conic varies from a parabola to a circle. It is to be noticed that, if the eccentricity is small, the particle falls in toward the center by an amount about twice as great as that required to carry it to the center in N revolutions if it moved in by the same amount during each revolution. This need not be perplexing, for according to the formula the particle is always falling in by a certain percentage of the distance remaining (not of the original total distance). It therefore appears necessary that the particle should fall in by an amount which at any rate for orbits of small eccentricity is decreasing from revolution to revolution. In fact if the increment of the radius vector be found to another approximation, it appears that such is the case.*

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* We have assumed tacitly that $L > 0$, that is, that the attraction is greater than the pressure of light.

As the eccentricity of the osculating ellipse seems to be of importance, it appears worth while to calculate the change of eccentricity during a revolution. Formula (13) shows that if the particle starts at an apse, it will be at an apse after a complete revolution provided we disregard the effect of the last approximation. This fact will materially simplify the discussion of the eccentricity. It enables us to write the equation of the osculating ellipse after one revolution as

$$u = \left(u_0 + \Delta u_0 - \frac{L}{(\phi_0 - 2\pi)^2}\right) \cos \xi + \frac{L}{(\phi_0 - 2\pi)^2}$$

On calculating the eccentricity of this ellipse by the usual formula we find

$$(15) \qquad \Delta e = -\frac{7}{2}\frac{e}{N}.$$

Hence it appears that the ellipse is always losing eccentricity. To speak roughly one may say that the orbit is losing its eccentricity from about two to about fifteen times as fast as it loses its perihelion distance according as the orbit is approximately circular or approximately parabolic.

Equation (13) shows that although there is not much change in the angular position of the apse, there is some. Let us assume that the apse advances by the amount ϵ, so that if $\xi = 0$ is a root of u', then $\xi = -2\pi - \epsilon$ is the next but one. It has been seen that ϵ is small of the order $1/N^2$. The second term of the first approximation gives a term in ϵ. All the terms of the second approximation are too small to contribute anything. The first and fifth terms of the third approximation contribute quantities independent of ϵ. The equation for ϵ therefore is

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* If I understand Poynting aright he states in *Nature*, loc. cit. p. 92, second column, that the particle will fall in farther during the succeeding revolutions than during the preceding ones. As Poynting very seldom makes a mistake, I have given the matter a great deal of attention but fail to find any error in my work. Moreover Poynting, in his earlier paper, obtains a formula which is interpretable in much the same way as mine. Unfortunately I cannot find that he has printed any mathematical treatment of the subject since then.

$$\left(u_0 - \frac{L}{\phi_0^2}\right)\epsilon = \frac{1}{\phi_0^2}\left[\frac{9L}{\phi_0^2}\,2\pi + \frac{3}{8}\left(u_0 - \frac{L}{\phi_0^2}\right)2\pi\right].$$

Hence

$$\epsilon = \frac{1}{2\pi N^2}\,\frac{1}{e}\left(9 + \frac{3}{8}e\right) = \frac{360}{\phi_0^2}\,\frac{1}{e}\left(9 + \frac{3}{8}e\right) \text{ degrees},$$

and there is a slight advance of the apse in the direction of motion. If the eccentricity is very small, that is, small of an order approaching $1/N^2$ the apse may advance over a considerable angle during a revolution.*

So much for the discussion of what happens in the vicinity of the start. We are now in a position to take up more general considerations. From the behavior of the apse investigated above, it should appear that as ϕ decreases from its initial value ϕ_0, the apses draw further and further apart; and conversely if we follow the motion backward allowing ϕ to increase, the apses will follow each other at intervals diminishing toward the limit π. Now this is what we should obtain on general considerations applied to formula (11). For the apses correspond to roots of u', that is, to roots of the expression

$$CJ_1(\phi) + DZ_1(\phi) - \frac{2L}{\phi^5}.$$

Here C and D are definite constants. To be sure they are very small in the cases we have been considering, in fact of the order $\phi_0^{-7/2}$. Nevertheless for values of ϕ sufficiently large, the third term, $-2L/\phi^5$, may be neglected relatively to the first two. Then the roots of u' may be investigated by the usual methods employed in discussing the roots of Bessel functions,† and we shall thereby obtain precisely the result concerning the advance of the apse that we have already obtained from our computations. As the value of u' becomes infinite when $\phi = 0$, it is clear that u' must have a definite smallest root corresponding to the last apse which the orbit possesses, after which it coils in around the attracting center with an essentially spiral form. It is not impossible to get a very rough estimate of where this last root occurs. For there can be no roots for values of ϕ so small that the third term above is larger than the other two. If we compare merely the magnitudes of ϕ, ϕ_0, C, and D, we find that the condition is fulfilled when $\phi = \phi_0^{7/9}$. To get any more accurate estimation seems not to be worth while.

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* It is, however, by no means certain that the orbit would keep its approximately conical nature for such small values of e: it might already have become spiral. See infra.

† Gray and Mathews, chap. 5.

Similar considerations may be applied to formula (10) to discuss the roots of u. If the test suggested above be used, it appears that the roots of u are likely to be found for values of ϕ of the same order of magnitude as the initial value ϕ_0. That this is correct follows from the approximate formula for u. Here, if the eccentricity e is greater then unity, there are roots of u in the immediate neighborhood of ϕ_0. If e is considerably less than unity, the motion of the particle may be followed back through a very large number of revolutions before the eccentricity shall have increased to unity. Nevertheless the rate of increase of e as given by (15) is tolerably fast, and it would be safe* to say that the first root of u would appear before ϕ had increased to many times ϕ_0. Now whenever u has a root, the particle has retreated to infinity. Hence it follows that not only is the number of revolutions which a particle can make inward toward the center a finite number N, but *the total number of revolutions which a particle can make in infinite time about an attracting luminous center is finite.* This result is so different from that of the ordinary theory of attraction as to be very striking. In fact, it could hardly be credited if the analysis had not assured its correctness. For values of ϕ greater than the first root of u the physical interpretation is simply that the particle is describing a hyperbolic type of path. As ϕ gets larger and larger the roots of u become separated by intervals equal more and more nearly to π. In other words the path is becoming more and more nearly straight. The analytical deductions, therefore, are seen to fulfil in every respect the conditions anticipated by the physical intuition.

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* In fact if we express C and D in terms of e_0 and if we assume that the particle starts at an apse where ϕ_0 has such a value that $Z_2(\phi_0) = 0$, then $J_1(\phi_0) = 0$ and the equation for roots of u may be written roughly in the form

$$\frac{1}{\phi_0}\frac{L}{\phi_0^2}e_0 Z_1(\phi_0)J_1(\phi) = \frac{L}{\phi^4}, \qquad Z_1(\phi_0) = \left(\frac{\pi}{2\phi_0}\right)^{1/2}$$

Hence we may look for roots of u in the neighborhood of

$$\phi^{7/2} = \frac{1}{e_0}\phi_0^{7/2}, \qquad \phi = e_0^{-2/7}\phi_0.$$

In closing we may take the time to obtain numerically some of the quantities which we have been discussing. The case of particular interest is that of a particle revolving about the sun. The following data may be assumed:

velocity of light, $V = 3 \times 10^{10}$
distance of sun from earth $= 493 \times V$
acceleration of earth in orbit $= 59 \times 10^{-2}$
pressure of sunlight at earth's distance $= 59 \times 10^{-6}$.

All these quantities are in the C. G. S. system. The constant C in equation (1) has the value

$$C = 59 \times 10^{-2} \times 493^{-2}\, V^2 = 1{\cdot}29 \times 10^{26}.$$

The constant K may be determined in terms of the radius a and the density ρ of the particle as

$$K : C = 59 \times 10^{-6}\, \pi a^2 : \tfrac{4}{3}\pi \rho a^3 59 \times 10^{-2}, \quad K = \tfrac{3}{4}\rho^{-1} a^{-1} 10^{-4} C.$$

If the particle has a density about equal to that of the earth and a radius of about 0·01cm, then $K = 1{\cdot}76 \times 10^{23}$. It should be noted that K is increased by a decrease in the radius or density, and that the value of K here obtained is certainly small for such particles as meteoric dust finely divided. Suppose that the initial double areal velocity is 45×10^{18} which is roughly that of the earth. Then if a particle be started at an apse at the earth's distance from the sun,

$$\phi_0 = 2{\cdot}3 \times 10^7, \quad N = 3{\cdot}66 \times 10^6, \quad e = 0{\cdot}052.$$

Hence the distance of fall from an apse will be 80·6km during the first revolution. The change of eccentricity will be about one millionth of its value. The apse will advance about one half of one millionth of a second of arc. It is very doubtful whether the particle could make so many as 10^7 revolutions altogether. If the particle had a radius of one tenth the amount above assumed, the fall would be ten times as great and so would the diminution of the eccentricity; the rotation of the apse, however, would be one hundred times as great as it was before.

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